How to solve quadratics and teach it well

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Motivation

When solving a quadratic

$$ax^2 + bx + c = 0$$

students are often taught to memorize and use the "quadratic formula":

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

But memorizing this does nothing good for the student and is lazy teaching. A more complete discussion of the perils of forcing memorization about this can be found here.

In this document, we provide a step-by-step guide for the student and the instructor on how to solve quadratics. Instructors can use this document to modify or supplement how they teach this topic. Students can use this document as an additional educational resource. Remember that this is on paper and for the uninitiated this may seem confusing. If you are having difficulty, contact me at help@mathmisery.com or leave a message on the Math Misery blog.

Solving quadratics

Let's say, we are interested in solving $x^2 + 6x - 7 = 0$. How would we do this? Well, let's keep this problem in the back of our mind, and set it aside for a moment. That will be our main goal. To get there Let's pick something a bit simpler.

Let's say we want to solve $(x + 3)^2 = 0$. How would we solve this? There are a few ways. We could just recognize that $x = -3$ sets the left-hand side equal to zero which matches the right-hand side and therefore is a solution. Another way, is to just go through the algebra.

1. Solve for x in $(x + 3)^2 = 0$
2. Take the square root on both sides giving $(x + 3) = \pm\sqrt{0}$
3. Subtract 3 from both sides giving $x = -3 \pm \sqrt{0}$ and since $\sqrt{0} = 0$, we have our solution: $x = -3$

Ok, that was easy. Next, let's say we want to solve $(x + 3)^2 - 16 = 0$. What now? Again, follow the algebra.

1. Solve for x in $(x + 3)^2 - 16 = 0$
2. Add 16 to both sides giving $(x + 3)^2 = 16$

3. Now this looks very similar to the problem above. So take square roots to get $(x+3) = \pm\sqrt{16}$
4. Subtract 3 from both sides to get $x = -3 \pm \sqrt{16}$ and since $\sqrt{16} = 4$, we have our solution**s**: $x = \{-3+4, -3-4\}$ or $x = \{-1, -7\}$

So far so good? Now, let's take a look at $(x+3)^2 - 16$ by itself. What happens if we expand this? Here are the steps:

$$\begin{aligned} (x+3)^2 - 16 &= (x+3)(x+3) - 16 \\ &= (x^2 + 3x + 3x + 9) - 16 \\ &= (x^2 + 6x + 9) - 16 \\ &= x^2 + 6x - 7 \end{aligned}$$

Hey! That looks like the left-hand side of the original problem at the beginning of this section. How did that happen? So let's take a closer, but somewhat generalized look at what's going on. Let's see what happens when we expand $(x+p)^2 + q$ where p, q are real numbers. In the above example, $p = 3, q = -16$.

$$\begin{aligned} (x+p)^2 + q &= (x+p)(x+p) + q \\ &= (x^2 + px + px + p^2) + q \\ &= x^2 + 2px + (p^2 + q) \end{aligned}$$

Now, this may look a bit messy, but it's actually quite nice if we step back for a second. Let's go back to the left-hand side of our original problem $x^2 + 6x - 7$ and look at what we just were able to put together for a general case. If we equate these two we have $x^2 + 6x - 7 = x^2 + 2px + (p^2 + q)$. Notice that on both sides there are x^2 terms (quadratic terms), x terms (linear terms), and constants (-7 on the left and $p^2 + q$ on the right). So if we equate "like terms" we get:

$$\begin{aligned} x^2 &= x^2 \\ 6x &= 2px \\ -7 &= p^2 + q \end{aligned}$$

There is nothing to do with the x^2 terms since they equate. So we can leave them alone. Now, $6x = 2px$ means that p must equal 3. Since $p = 3$, that means $-7 = 3^2 + q \implies -7 = 9 + q \implies -16 = q$. Putting all that together we have $(x+p)^2 + q = (x+3)^2 + -16$. And this method is called "completing the square".

The student may find it helpful to ponder that phrase for a little bit. Why is it called "completing the square"? What is being completed? What is a "square" since there is does not seem to be any mention of geometry here? (As an aside, there *is* a geometric interpretation, but out of scope for this document. Here's a quick, dirty, and loose explanation: $(x+p)^2$ represents the area of a square of length $(x+p)$. The constant q represents excess / shortfall in area (depending on if q is positive or not). Thus, if there were some rectangle, what's a "representative" square and how much area do we have to add or take away to make it "fit" with the rectangle. Remember, this is a quick, dirty, and loose explanation. I'm doing some hand-waving, but the student should think about this.)

A worked out example

Solve $x^2 + 10x + 8 = 0$.

1. Well if we could write $x^2 + 10x + 8 = 0$ as $(x + p)^2 + q = 0$ then we could just do the algebra to solve
2. We already did the hard work of showing that $(x + p)^2 + q = x^2 + 2px + (p^2 + q)$. So that must mean that $10x = 2px$ and $8 = p^2 + q$
3. $10x = 2px \implies p = 5$
4. Since $p = 5$ then $8 = 5^2 + q \implies -17 = q$
5. So $x^2 + 10x + 8 = (x + 5)^2 - 17$
6. Therefore, $(x + 5)^2 - 17 = 0 \implies x = 5 \pm \sqrt{17}$

Summary

Instructors: I hope that you are a little more convinced, if you already weren't convinced, that there is no need to asks students to memorize the quadratic formula. Nor is there any reason to have students memorize the method for completing the square. Providing logical step-by-step explanations and getting students to speak about their work, step-by-step, will be far more helpful for them than having them recite formulas. Also, I hope that you will be able to add to your teaching repertoire from this article. There are more things to cover. Explain to students a further generalization. How do we solve $ax^2 + bx + c = 0$? I only explained $a = 1$. But it should be straightforward to see that all that needs to be done is to divide the equation by a and the problem reverts to something already solved.

Students: The first time around may not be clear. That's why this is written up. Reread it! Ask your instructor for help! Contact me (see the top of this document) if you need anything clarified.